Analytic multivariate generating function for random multiplicative cascade processes[*]

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We have found an analytic expression for the multivariate generating function governing all n-point statistics of random multiplicative cascade processes. The variable appropriate for this generating function is the logarithm of the energy density, $\ln \epsilon$, rather than ϵ itself. All cumulant statistics become sums over derivatives of "branching generating functions" which are Laplace transforms of the splitting functions and completely determine the cascade process. We show that the branching generating function is a generalization of the multifractal mass exponents. Two simple models from fully developed turbulence illustrate the new formalism.

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Multifractals have become a popular tool especially in fully developed turbulence to analyze the intermittent fluctuations occurring in the energy dissipation field [1,2]. They have also been applied to characterize the phase space structure of strange attractors in chaotic dynamical systems [3], diffusion limited aggregation [4], high-energetic multiparticle dynamics [5], geophysics and self-organized criticality [6], to name just a few.

Among the simplest examples of multifractal processes are random multiplicative cascade processes in general and the multiplicative binomial process [7,8] in particular. Both the multifractal approach and large deviation theory analyse these processes in terms of one-point statistics. However, because different cascade processes like, for example, the p-model [8] and the α-model [9] are indistinguishable in this framework, such one-point statistics appear to be too restrictive. In order to discriminate among these and other suggested models, a generalisation of the multifractal approach to n-point statistics is thus called for. The analytic multivariate generating function presented in this letter is, we believe, the appropriate generalisation.

Two-point statistics of random multiplicative cascade processes were previously discussed in Refs. [10,11]. An approach to calculate the n-point statistics, or in other words the spatial correlations to arbitrary order, within these models has been presented in Refs. [12,13]: there, the multivariate generating function of the spatial correlations was constructed iteratively from a backward evolution equation, leading to a recursive derivation of spatial correlations. While in the latter approach, models like the p- and α-model become clearly distinguishable, the recursive structure of the multivariate generating function appears from a mathematical perspective to be less than elegant: an analytic solution is much more desireable. The realization of this goal is the subject of this letter.

In order to derive the analytic expression for the multivariate generating function of random multiplicative binary cascade processes, we first need to find the appropriate variables and to introduce a convenient labelling. The cascade prescription is as follows: A given initial interval of length 1 and unit energy density successively splits into 2, 4, 8, ... subintervals ("bins"). For the latter, we employ a binary labeling: after j cascade steps (interval splittings), a specific subinterval is characterized by the sequence $k_1 k_2 \cdots k_j$ with each k_i being either 0 or 1. The bin with the label $00 \cdots 0$ is the leftmost one, whereas the label $11 \cdots 1$ belongs to the bin on the far right. The energy density $\epsilon_{k_1 \cdots k_j}^{(j)}$ belonging to a j-th generation bin is redistributed nonuniformly onto the two intervals of the next generation: $\epsilon_{k_1 \cdots k_j 0}^{(j+1)} = q_{k_1 \cdots k_j 0}^{(j+1)} \epsilon_{k_1 \cdots k_j}^{(j)}$, and $\epsilon_{k_1 \cdots k_j 1}^{(j+1)} = q_{k_1 \cdots k_j 1}^{(j+1)} \epsilon_{k_1 \cdots k_j}^{(j)}$. The two multipliers $q_{k_1 \cdots k_j 0}$ and $q_{k_1 \cdots k_j 1}$ are drawn from a probability density $p(q_{k_1 \cdots k_j 0}, q_{k_1 \cdots k_j 1})$ which we call the splitting function.

After J cascade steps, the energy density belonging to bin $(k_1 k_2 \cdots k_J)$ is given by the product of multipliers over all previous generations

$$\epsilon_{k_1 \cdots k_J}^{(J)} = q_{k_1}^{(1)} q_{k_1 k_2}^{(2)} \cdots q_{k_1 \cdots k_J}^{(J)}. \tag{1}$$

For the logarithm of the energy density this relation becomes of course additive; this observation is crucial for the subsequent derivation.

The structure of a specific random multiplicative cascade model is completely determined by the splitting function $p(q_0, q_1)$ which determines the distribution of energy during each split. Hence, $p(q_0, q_1)$ also completely determines the multivariate statistics of a given cascade. The joint probability density $\tilde{p}(\ln \epsilon_{0 \cdots 00}^{(J)}, \ldots, \ln \epsilon_{1 \cdots 11}^{(J)})$ to find at the same time the logarithm of $\epsilon_{0 \cdots 00}^{(J)}$ in the bin with label $(0 \cdots 00)$, the value $\ln \epsilon_{0 \cdots 01}^{(J)}$ in bin $(0 \cdots 01)$, ..., and $\ln \epsilon_{1 \cdots 11}^{(J)}$ in bin $(1 \cdots 11)$ can therefore be expressed fully in terms of the splitting functions at each branching:

$$\tilde{p}(\ln \epsilon_{0 \cdots 0}^{(J)}, \ldots, \ln \epsilon_{1 \cdots 1}^{(J)}) =$$

$$\int \Big[\prod_{j=1}^{J} \prod_{\substack{k_1,\ldots,\\ k_{j-1}=0}}^{1} dq^{(j)}_{k_1\cdots k_{j-1}0} dq^{(j)}_{k_1\cdots k_{j-1}1} p(q^{(j)}_{k_1\cdots k_{j-1}0}, q^{(j)}_{k_1\cdots k_{j-1}1})\Big]$$

$$\times \Big[\prod_{\substack{k_1,\ldots,\\ k_J=0}}^{1} \delta\Big(\ln \epsilon^{(J)}_{k_1\cdots k_J} - \sum_{j=1}^{J} \ln q^{(j)}_{k_1\cdots k_j}\Big)\Big]. \qquad (2)$$

This probability density can be converted into a multivariate generating function for cumulants in $\ln \epsilon$. With the definition

$$K[\lambda^{(J)}_{0\cdots0},\ldots,\lambda^{(J)}_{1\cdots1}] = \ln\Big\langle \exp\Big(\sum_{\substack{k_1,\ldots,\\ k_J=0}}^{1} \lambda^{(J)}_{k_1\cdots k_J} \ln \epsilon^{(J)}_{k_1\cdots k_J}\Big)\Big\rangle$$

$$= \ln\Big[\int \Big(\prod_{\substack{k_1,\ldots,\\ k_J=0}}^{1} d(\ln \epsilon^{(J)}_{k_1\cdots k_J})\Big) p(\ln \epsilon^{(J)}_{0\cdots0},\ldots,\ln \epsilon^{(J)}_{1\cdots1})$$

$$\times \exp\Big(\sum_{\substack{k_1,\ldots,\\ k_J=0}}^{1} \lambda^{(J)}_{k_1\cdots k_J} \ln \epsilon^{(J)}_{k_1\cdots k_J}\Big)\Big] \qquad (3)$$

we find, after defining

$$\lambda^{(j)}_{k_1\cdots k_j} = \sum_{k_{j+1},\ldots,k_J=0}^{1} \lambda^{(J)}_{k_1\cdots k_j k_{j+1}\cdots k_J}, \qquad (4)$$

rearranging the terms in the exponent of (3), namely

$$\sum_{k_1,\ldots,k_J} \lambda^{(J)}_{k_1\cdots k_J} \ln \epsilon^{(J)}_{k_1\cdots k_J} = \sum_{j=1}^{J} \sum_{k_1,\ldots,k_j} \lambda^{(j)}_{k_1\cdots k_j} \ln q^{(j)}_{k_1\cdots k_j}$$

$$= \sum_{j=1}^{J} \sum_{\substack{k_1,\ldots,\\ k_{j-1}}} \Big(\lambda^{(j)}_{k_1\cdots k_{j-1}0} \ln q^{(j)}_{k_1\cdots k_{j-1}0} + \lambda^{(j)}_{k_1\cdots k_{j-1}1} \ln q^{(j)}_{k_1\cdots k_{j-1}1}\Big), \quad (5)$$

and inserting (2) into (3), that the 2^J-fold integral factorizes, so that

$$K[\lambda^{(J)}_{0\cdots0},\ldots,\lambda^{(J)}_{1\cdots1}] = \sum_{j=1}^{J} \sum_{\substack{k_1,\ldots,\\ k_{j-1}=0}}^{1} Q[\lambda^{(j)}_{k_1\cdots k_{j-1}0}, \lambda^{(j)}_{k_1\cdots k_{j-1}1}], \quad (6)$$

where

$$Q[\lambda_0, \lambda_1] = \ln\Big[\int dq_0\, dq_1\, p(q_0, q_1) e^{\lambda_0 \ln q_0 + \lambda_1 \ln q_1}\Big] \quad (7)$$

is the "branching generating function" (b.g.f.) for cumulants, which governs the behavior of the entire cascade. Eqs. (6)–(7) represent the long-sought analytic expression for multiplicative cascades: because the multivariate cumulant generating function K is the sum of all branching generating functions Q, one for every branching, each b.g.f. can be solved separately and analytically.

Multivariate cumulants likewise become sums over cumulants of the individual branching points. Using the notation $\boldsymbol{\kappa} \equiv (k_1 \cdots k_J)$ in the subscripts, the multivariate cumulant of order n is found through

$$C_{\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_n} = \frac{\partial^n K[\lambda^{(J)}]}{\partial \lambda_{\boldsymbol{\kappa}_1} \cdots \partial \lambda_{\boldsymbol{\kappa}_n}}\Big|_{\lambda^{(J)}=0}. \quad (8)$$

The first three multivariate cumulants are

$$C_{\boldsymbol{\kappa}_1} = \langle \ln \epsilon_{\boldsymbol{\kappa}_1}\rangle_c = \langle \ln \epsilon_{\boldsymbol{\kappa}_1}\rangle, \quad (9)$$

$$C_{\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2} = \langle \ln \epsilon_{\boldsymbol{\kappa}_1} \ln \epsilon_{\boldsymbol{\kappa}_2}\rangle_c$$

$$= \langle \ln \epsilon_{\boldsymbol{\kappa}_1} \ln \epsilon_{\boldsymbol{\kappa}_2}\rangle - \langle \ln \epsilon_{\boldsymbol{\kappa}_1}\rangle\langle \ln \epsilon_{\boldsymbol{\kappa}_2}\rangle, \quad (10)$$

$$C_{\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2,\boldsymbol{\kappa}_3} = \langle \ln \epsilon_{\boldsymbol{\kappa}_1} \ln \epsilon_{\boldsymbol{\kappa}_2} \ln \epsilon_{\boldsymbol{\kappa}_3}\rangle_c$$

$$= \langle \ln \epsilon_{\boldsymbol{\kappa}_1} \ln \epsilon_{\boldsymbol{\kappa}_2} \ln \epsilon_{\boldsymbol{\kappa}_3}\rangle$$

$$- \sum_{\text{perm}} \langle \ln \epsilon_{\boldsymbol{\kappa}_1} \ln \epsilon_{\boldsymbol{\kappa}_2}\rangle\langle \ln \epsilon_{\boldsymbol{\kappa}_3}\rangle$$

$$+ 2\langle \ln \epsilon_{\boldsymbol{\kappa}_1}\rangle\langle \ln \epsilon_{\boldsymbol{\kappa}_2}\rangle\langle \ln \epsilon_{\boldsymbol{\kappa}_3}\rangle. \quad (11)$$

These and higher-order cumulants can be found through appropriate derivatives of the b.g.f. using (6). If, for example, the same b.g.f. is used for all branchings, and the splitting function is symmetric, $p(q_0, q_1) = p(q_1, q_0)$, the two-point cumulant is

$$C_{\boldsymbol{\kappa},\boldsymbol{\kappa}'} = \Big(\sum_{j=1}^{J} \delta_{k_1 k_1'} \cdots \delta_{k_j k_j'}\Big) \frac{\partial^2 Q[\lambda_0, \lambda_1]}{\partial \lambda_0^2}\Big|_{\lambda=0} \quad (12)$$

$$+ \Big(1 - \delta_{k_1 k_1'} \cdots \delta_{k_J k_J'}\Big) \frac{\partial^2 Q[\lambda_0, \lambda_1]}{\partial \lambda_0 \partial \lambda_1}\Big|_{\lambda=0}.$$

For the p-model with a splitting function $p(q_0, q_1) = \frac{1}{2}[\delta(q_0 - (1+\alpha)) + \delta(q_0 - (1-\alpha))]\delta(q_0 + q_1 - 2)$, the second-order cumulant correlation density is shown in Fig. 1.

We should point out the scope and limitations of the solution (6). Clearly, the generating function in terms of $\ln \epsilon$ is applicable to any functional form of the splitting function or b.g.f. It does not depend on the number of branches either: trivariate or even higher-variate splitting functions can be implemented. Due to the additive nature of the b.g.f.'s, the splitting functions can differ from generation to generation and even from branch to branch. The only (and important) precondition for the applicability of Eq. (6) is that the splitting variables q of every branching must be independent of all others[1].

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[1] Nelkin and Stolovitzky [14] argue that the dependence of experimental distributions of splitting variables (multiplier distributions) on the position of the subinterval [15] suggests that the splitting variables are in fact not statistically independent; see also Ref. [16]. The effects of this deviation from statistical independence on the present formalism, and to what extent it matters, remain to be investigated.
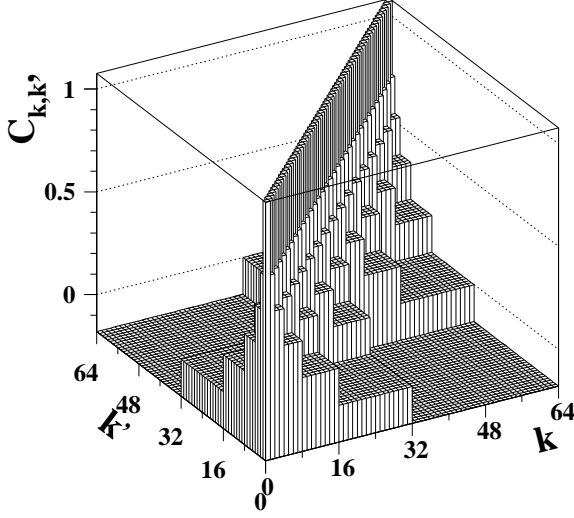
Figure 1:
Second-order cumulant $C_{k,k'} \equiv C_{(k_1 \cdots k_J),(k'_1 \cdots k'_J)}$ for the p-model, where $k = 1 + \sum_{j=1}^{J} k_j 2^{J-j}$ and $k' = 1 + \sum_{j=1}^{J} k'_j 2^{J-j}$. Parameters have been set to $J = 6$ and $\alpha = 0.4$.

Moments of the J-generation cascade are found directly from the moment generating function $Z[\lambda^{(J)}] = \exp(K[\lambda^{(J)}])$. From Eq. (6), it is obvious that Z factorizes into moment generating functions of the individual branchings.

The key input into the analytical expression (6) is the branching generating function $Q[\lambda_0, \lambda_1]$ of Eq. (7). Its properties uniquely fix the spatial correlations of the binary random multiplicative cascade model. For the p-model, we get

$$Q[\lambda_0, \lambda_1]_{p-\text{model}} = \tfrac{1}{2}(\lambda_0 + \lambda_1)\ln(1 - \alpha^2) \qquad (13)$$
$$+ \ln\left\{\cosh\left[\frac{1}{2}(\lambda_0 - \lambda_1)\ln\left(\frac{1+\alpha}{1-\alpha}\right)\right]\right\}.$$

The symmetric α-model is similar to the p-model except that it does not conserve energy in a cascade splitting. Given its splitting function, $p(q_0, q_1) = \frac{1}{4}\prod_{k=0}^{1}[\delta(q_k - (1+\alpha)) + \delta(q_k - (1-\alpha))]$, its b.g.f. then reads

$$Q[\lambda_0, \lambda_1]_{\alpha-\text{model}} = \tfrac{1}{2}(\lambda_0 + \lambda_1)\ln(1 - \alpha^2) \qquad (14)$$
$$+ \ln\left\{\cosh\left[\frac{1}{2}\lambda_0\ln\left(\frac{1+\alpha}{1-\alpha}\right)\right]\right\}$$
$$+ \ln\left\{\cosh\left[\frac{1}{2}\lambda_1\ln\left(\frac{1+\alpha}{1-\alpha}\right)\right]\right\}$$

which clearly differs from the b.g.f. (13) for the p-model. Fig. 2 compares the two branching generating functions. Note that for $\lambda_0 = 0$ or $\lambda_1 = 0$ the two expressions (13) and (14) become identical; consequently, the one-point cumulants

$$\left.\frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^n}\right|_{\lambda=0} = \langle(\ln q_0)^n\rangle_c \qquad (15)$$

of the p- and α-model are also identical. This is the reason why, in a multifractal approach, the two models look the same asymptotically. To see differences between the two models, one must go to the two-point cumulants

$$\left.\frac{\partial^n Q[\lambda_0, \lambda_1]}{\partial \lambda_0^{n_1} \partial \lambda_1^{n-n_1}}\right|_{\lambda=0} = \left\langle(\ln q_0)^{n_1}(\ln q_1)^{n-n_1}\right\rangle_c. \qquad (16)$$

Within the p-model, the latter are nonzero for even n and zero for odd $n \geq 3$. Within the α-model, by contrast, all two-point cumulants vanish since its splitting function factorizes: $p(q_0, q_1) = p(q_0)p(q_1)$. We hence see that the two-point cumulant moments are sensitive to the violation of energy conservation in the splitting function.

For a binary random multiplicative cascade process to qualify as a true multifractal process, the splitting function needs to conserve energy, i.e. $p(q_0, q_1) = p(q_0)\delta(q_0 + q_1 - 2)$; see Ref. [13] for a clarification of this point. If energy is indeed conserved, it is possible to link the multifractal mass exponents $\tau(q)$ to the b.g.f. $Q[\lambda_0, \lambda_1]$. Setting $\lambda_1 = 0$ and, for simplicity, considering the case of a symmetric splitting function $p(q_0, q_1) = p(q_1, q_0)$, Eq. (7) becomes[2]

$$Q[\lambda_0, \lambda_1 = 0] = \ln\left[\int dq_0\, p(q_0)q_0^{\lambda_0}\right] = \ln\langle q_0^{\lambda_0}\rangle$$
$$= \ln 2\Big[(\lambda_0 - 1) + \tau(\lambda_0)\Big]; \qquad (17)$$

note that for clarity we have written $\tau(\lambda_0)$ instead of the more familiar notation $\tau(q)$. Hence, for an energy-conserving splitting function, its multifractal mass exponents follow from $Q[\lambda_0, \lambda_1]$. The reverse need not be true, though.

Also, for the more general case where energy is not conserved in the splitting function, the multifractal mass exponents $\tau(q)$ cannot be deduced in a clean fashion and contain less information than the b.g.f. This means that, for binary multiplicative cascades, $Q[\lambda_0, \lambda_1]$ can be understood as the natural generalization of the multifractal mass exponents.

For a one-dimensional cut through the three-dimensional energy dissipation field in fully developed turbulence, it is most likely that energy is not conserved along the cut. Hence, the splitting function will not conserve energy and cannot be identified with the scale-invariant multiplier distributions [16]. Here it would be interesting to find a procedure to infer the proper splitting function from data. Given that the experimentally

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[2] The univariate version of relation (17) was previously discussed by Novikov [17] in connection with the statistics of generalised multipliers, the so-called breakdown coefficients.

measurable cumulants in $\ln \epsilon_{k_1 \cdots k_J}^{(J)}$ are n-fold derivatives of Q, the latter can in principle be reconstructed from the former. With the help of Eq. (7), the b.g.f. can then be inverted into the splitting function via a two-dimensional inverse Laplace transformation,

$$\int_0^\infty dx\,dy\, p(2e^{-x}, 2e^{-y})\, e^{-(\lambda_0+1)x-(\lambda_1+1)y}$$
$$= e^{Q[\lambda_0,\lambda_1]-(\lambda_0+\lambda_1+2)\ln 2}. \tag{18}$$
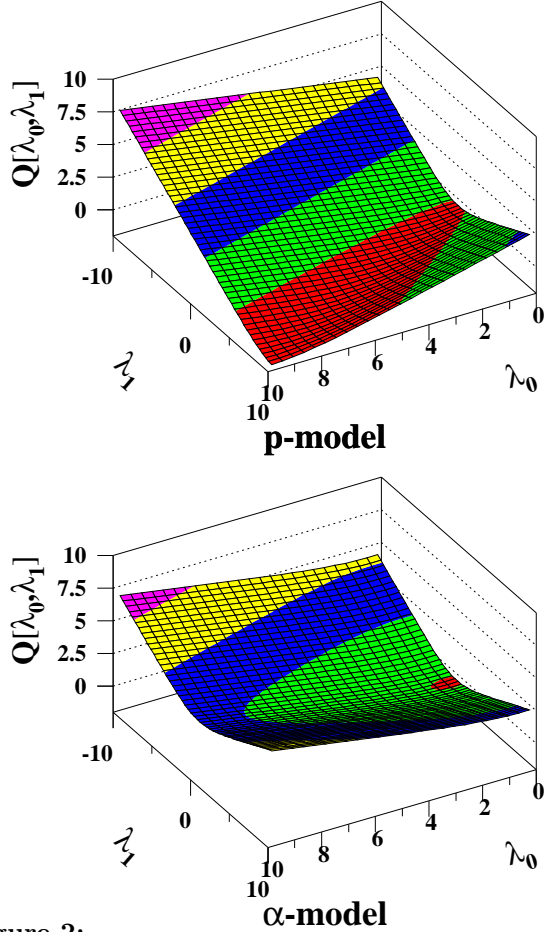
p-model

α-model

Figure 2:
Branching generating function $Q[\lambda_0, \lambda_1]$ for the p- and α-models for $\alpha = 0.4$, shown for $0 \leq \lambda_0 \leq 10$; $-10 \leq \lambda_1 \leq 10$. The intersection of Q with the $\lambda_0=0$ plane is related to the multifractal $\tau(\lambda_0)$ curve through Eq. (17). Note that these curves are identical for the $\alpha-$ and p-models, while otherwise the b.g.f.'s differ significantly.

Of course, when confronted with real turbulence data, the proposed inversion will not be this straightforward as the problems of homogeneity [18] and statistical dependence of multipliers [14–16] have to be taken into account.

Within the above limitations, we envisage many and diverse applications of our analytic solution in many branches of physics. Besides fully-developed turbulence, the case of high-energetic multiparticle branching processes immediately comes to mind. For the latter, the α-

and p-models have already been used in this context as simulation toy models [19,20]. Implications in this and, for example, random multiplicative process calculations in large-scale structure formation in the universe [21] remain to be explored.

* Dedicated to the memory of Peter Carruthers, teacher and friend.
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